

# Fixing the fixed-point system - Dynamic Renormalization Group revisited

**E Katzav**

Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, CNRS UMR 8550, 24 rue Lhomond, 75231 Paris Cedex 05, France.

E-mail: [eytan.katzav@lps.ens.fr](mailto:eytan.katzav@lps.ens.fr)

**Abstract.** In this paper a modified version of the Dynamic Renormalization Group (DRG) method is suggested in order to cope with inconsistent results obtained when applying it to a continuous family of one-dimensional models. The key observation is that the correct fixed-point dynamical system has to be identified during the analysis in order to account for all the relevant terms that are generated under renormalization. An application of this approach to the nonlocal Kardar-Parisi-Zhang equation resolves the known problems in one-dimension. Namely, obviously problematic predictions are eliminated and the existing exact analytic results are recovered.

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Fluctuating surfaces appear in a wide variety of physical situations and have been of great interest in the last two decades [1, 2, 3]. These and other systems far from thermal equilibrium pose a major challenge in contemporary statistical physics. Behavior out-of-equilibrium is far richer than at equilibrium, and many intriguing scaling phenomena, such as self-organized criticality [4], or phase transitions between non-equilibrium stationary states [1], have been observed for long. However, despite the considerable achievements, the theoretical comprehension of non-equilibrium phenomena remains much poorer than our understanding of equilibrium phenomena.

The renormalization group (RG), proven useful to explain universality in equilibrium continuous phase transitions, has also allowed some progress in understanding systems out-of-equilibrium. Nevertheless, in many cases the information RG analysis offers is not complete and limited to a certain range of dimensions. A classical example is the Kardar-Parisi-Zhang (KPZ) equation [3] where the Dynamic Renormalization Group (DRG) approach agrees with the analytic exact result in one dimension [1] but unable to provide results for the strong coupling phase in higher dimension. This clearly indicates that internal problems exist in the DRG calculation for  $d > 1$ . Actually, a remarkable result of Wiese [5] shows that the shortcoming of DRG in the KPZ system is not an artifact of a low order calculation (so called "one loop" calculation), but rather intrinsic to the method and extends to all orders. This situation motivated the development of other methods to deal with the KPZ system such as a scaling approach [6], Self-Consistent Expansion (SCE) [7], Mode-Coupling [8] and others that were able to provide predictions for the exponents in more than one-dimension.

A decade ago, a family of nonlocal growth models have been introduced in [9], known as the Nonlocal KPZ (NKPZ) equation, to account for nonlocal interactions in a system of deposited colloids, giving rise to roughness larger than the one predicted by the classical KPZ case. The authors studied the white noise case that was later generalized to spatially correlated noise in [10]. To be more specific, the equation they studied was

$$\frac{\partial h(\vec{r}, t)}{\partial t} = \nu \nabla^2 h(\vec{r}, t) + \frac{\lambda_\rho}{2} \int d^d r' \frac{\nabla h(\vec{r}, t) \cdot \nabla h(\vec{r}', t)}{|\vec{r} - \vec{r}'|^{d-\rho}} + \eta(\vec{r}, t), \quad (1)$$

where  $\eta(\vec{r}, t)$  is a noise-term modeling the fluctuation of the rate of deposition, which has a zero mean and is characterized by its second moment

$$\langle \eta(\vec{r}, t) \eta(\vec{r}', t') \rangle = 2D_0 |\vec{r} - \vec{r}'|^{2\sigma-d} \delta(t - t'), \quad (2)$$

where  $d$  is the substrate dimension and  $D_0$  specifies the noise amplitude. Both papers [9, 10] investigated this problem using Dynamic Renormalization Group (DRG), and derived a complex phase diagram. Focusing on the strong coupling solution (in the KPZ sense [1, 3]) both papers found

$$z = 2 + \frac{(d - 2 - 2\rho)(d - 2 - 3\rho)}{(3 + 2^{-\rho})d - 6 - 9\rho}, \quad (3)$$

where  $z$  is the dynamic exponent. The roughness exponent,  $\alpha$ , characterizing the long distance spatial behavior, is obtained using the modified Galilean scaling relation

$$\alpha + z = 2 - \rho.$$

Unfortunately, the DRG results for the exponents summarized in Eq. (3) above, were found to be inconsistent with an exact result available in 1D [12] predicting  $z = (3 - 3\rho)/2$  when  $\rho = 2\sigma$ . A more systematic study using the Self-Consistent Expansion [11] led to the hypothesis that DRG fails in recovering the exact one-dimensional result since it does not account for new modes of relaxation generated by the special nonlinearity. This suggests going back the old Renormalization idea of identifying the right fixed point dynamical system around which the expansion should be performed. The fixed point dynamical system is not necessarily of the same form as the original system, as is implicitly assumed by the standard DRG procedure.

In this paper a modification of the standard DRG procedure that goes along those lines is suggested. This approach makes DRG more flexible, and succeeds in recovering the exact result for the case of NKPZ. Not less important, this approach could be useful in implementing DRG in other situations where long-range interactions are present, such as those appearing in the context of wetting of an amorphous solid by a liquid [13, 14] and in in-plane tensile crack propagation in a disordered medium [15, 16]. The main motivation here is to make the first step towards extending the range of applicability of DRG in a field that suffers anyway from a lack of analytical tools, in order to allow further progress in systems out-of-equilibrium.

To understand the origin of the difficulty, consider the one loop DRG. The renormalization procedure is most succinctly described through the Fourier momentum  $q$  and frequency  $\omega$  modes, in terms of which Eq. (1) becomes

$$h(\vec{k}, \omega) = G_0(\vec{k}, \omega) \eta(\vec{k}, \omega) + \lambda_\rho N[h(\vec{k}, \omega)], \quad (4)$$

where  $G_0(\vec{k}, \omega)$  is the bare propagator given by  $G_0(\vec{k}, \omega) \equiv 1/(\nu_0 k^2 - i\omega)$ , and  $N[h(\vec{k}, \omega)]$  is a nonlinear functional of the height given by

$$N[h(\vec{k}, \omega)] = -\frac{1}{2} G_0(\vec{k}, \omega) k^{-\rho} \int \int \frac{d^d \vec{q} d\Omega}{(2\pi)^{d+1}} \vec{q} \cdot (\vec{k} - \vec{q}) h(\vec{q}, \Omega) h(\vec{k} - \vec{q}, \omega - \Omega), \quad (5)$$

The one loop expression for the dressed propagator defined by  $G(\vec{k}, \omega) \equiv h(\vec{k}, \omega) / \eta(\vec{k}, \omega)$  is given [1, 17] by

$$\begin{aligned} G(\vec{k}, \omega) &= G_0(\vec{k}, \omega) + 4 \left( -\frac{\lambda}{2} \right)^2 G_0^2(\vec{k}, \omega) \\ &\times \int \int \frac{d^d \vec{q} d\Omega}{(2\pi)^{d+1}} 2D_0 q^{-2\sigma} |\vec{k} - \vec{q}|^{-\rho} [\vec{q} \cdot (\vec{k} - \vec{q})] k^{-\rho} [(-\vec{q}) \cdot \vec{k}] \\ &\times G_0(\vec{k} - \vec{q}, \omega - \Omega) G_0(\vec{q}, \Omega) G_0(-\vec{q}, -\Omega), \end{aligned} \quad (6)$$

which, after some algebra (see appendix B in Ref. [1] for example) gives

$$G(\vec{k}, 0) = G_0(\vec{k}, 0) + \frac{\lambda^2 D_0}{\nu_0^2} G_0^2(\vec{k}, 0) k^{2-\rho} K_d \frac{d + \rho - 2 - 2\sigma}{4d} \int^\Lambda dq q^{d-3-2\sigma-\rho}, \quad (7)$$

where  $K_d \equiv S_d/(2\pi)^d$ , and  $S_d$  is the surface area of a  $d$ -dimensional unit sphere.

In local KPZ the last equation is used to calculate the renormalization of the surface tension  $\nu_0$ . However, a look at equation (7) highlights the problem. While the first term on the RHS scales as  $k^{-2}$  the second scales as  $k^{-2-\rho}$ . Here a distinction between three cases should be made: (a) When  $\rho < 0$  the correction term (proportional to  $\lambda^2$ ) is irrelevant compared to the first term in the limit of small momentum (i.e. in the limit of large scales). (b) When  $\rho = 0$  both terms have the same scaling dimension. This situation is actually the case in the classical KPZ equation (with correlated noise), which is well studied for example in Refs. [17, 18, 19]. And (c) when  $\rho > 0$  the correction is dominant over the first term. This means that in this situation the perturbative expansion produces more relevant terms than those originally present in the equation. More specifically a fractional Laplacian is produced under the renormalization. This implies that the fixed-point system in the space of dynamical systems is a-priori not the original model, and one needs to consider a more general form which contains such a term in the equation in the first place. Adding a  $\nu_1 k^{2-\rho}$  term by hand and going through the same process, a new (partially) dressed propagator  $G_1(\vec{k}, \omega) = 1/(\nu_1 k^{2-\rho} + i\omega)$  is obtained. Repeating the steps described above gives a  $2^{nd}$  order expansion for the full propagator, similar to Eq. (7)

$$G(\vec{k}, 0) = G_1(\vec{k}, 0) + \frac{\lambda^2 D_0}{\nu_1^2} G_1^2(\vec{k}, 0) k^{2-\rho} K_d \frac{d-2\sigma-2+2\rho}{4d} \int^\Lambda dq q^{d-3-2\sigma+\rho}, \quad (8)$$

This time, all the terms have the same scaling dimension so that the perturbative expansion is meaningful in the sense that higher order corrections are not more relevant than lower order ones. This allows to calculate the renormalization of the effective surface tension  $\tilde{\nu}_1$  when  $\rho > 0$ .

$$\tilde{\nu}_1 = \nu_1 \left[ 1 - \frac{\lambda^2 D_0}{\nu_1^3} \frac{d-2\sigma-2+2\rho}{4d} K_d \int^\Lambda dq q^{d-3-2\sigma+\rho} \right]. \quad (9)$$

Next, the renormalization of the noise term is calculated. The effective noise  $\tilde{D}$  is defined as the contraction of two terms according to

$$\langle h(\vec{k}, \omega) h(\vec{k}', \omega') \rangle = 2\tilde{D} G(\vec{k}, \omega) G(\vec{k}', \omega') \delta^d(\vec{k} + \vec{k}') \delta(\omega + \omega'). \quad (10)$$

The one loop expansion yields now

$$\begin{aligned} 2\tilde{D}k^{-2\sigma} &= 2D_0k^{-2\sigma} + 2(2D_0)^2 \left(-\frac{\lambda}{2}\right)^2 k^{-2\rho} \int \int \frac{d^d \vec{q} d\Omega}{(2\pi)^{d+1}} \\ &\times \left[ \vec{q} \cdot (\vec{k} - \vec{q}) \right]^2 q^{-2\sigma} |\vec{k} - \vec{q}|^{-2\sigma} \left| G_1(\vec{k} - \vec{q}, \omega - \Omega) \right|^2 |G_1(\vec{q}, \Omega)|^2. \end{aligned} \quad (11)$$

Notice that  $G_1(\vec{k}, \omega)$  was used in the last expression. In case  $\rho < 0$  this should be  $G_0(\vec{k}, \omega)$  as in standard DRG. Evaluating the integral in Eq. (11) one obtains

$$\tilde{D}k^{-2\sigma} = D_0k^{-2\sigma} + k^{-2\rho} \frac{\lambda^2 D_0^2}{\nu_1^3} \cdot \frac{K_d}{4} \int^\Lambda dq q^{d-3-4\sigma+3\rho} \quad (12)$$

(or  $\int^\Lambda dq q^{d-3-4\sigma}$  when  $\rho < 0$ ). As before, the behavior of this equation is complicated by the  $k$ -dependence of the correction term, and there are three options: (a) When  $\rho < \sigma$  the correction is irrelevant and the noise amplitude does not renormalize. (b) For  $\rho = \sigma$  the second term is of the same order as the first term, and therefore renormalizes the noise amplitude (the KPZ equation is an example of this case since there  $\rho = \sigma = 0$ ). And (c) for  $\rho > \sigma$  the correction term is more relevant than the first term. This means that in this situation the perturbative expansion produces an additional correlated noise which is more relevant than that originally present in the equation. This implies, just like for the propagator above, that the fixed-point system in the space of dynamical systems does not have the form of the original model, and a more general form with a new noise term  $D_1 k^{-2\rho}$  is considered. Doing the RG calculation from the beginning gives

$$\tilde{D}k^{-2\rho} = D_1 k^{-2\rho} \left[ 1 + \frac{\lambda^2 D_1}{\nu_1^3} \cdot \frac{K_d}{4} \int^\Lambda dq q^{d-3-\rho} \right]. \quad (13)$$

Last, the one-loop contribution to the vertex  $\lambda_\rho$  is calculated. Without getting into all the details, the final results is that the vertex does not renormalize to one-loop order  $\tilde{\lambda}_\rho = \lambda_\rho$ , since the structure of the perturbation theory is analytical in nature and cannot generate singular terms that renormalizes  $\lambda_\rho$ . This is simpler than the non renormalization of the vertex in the classical KPZ case (with  $\rho = 0$ ), where the correction is identically zero because of some exact cancelation of terms (see Fig. B.3(c) in [1]).

Following the standard rescaling procedure the following flow equations are obtained,

$$\begin{aligned} \frac{d\nu_0}{d\ell} &= \nu_0 (z - 2) & \rho < 0 \\ \frac{d\nu_1}{d\ell} &= \nu_1 \left( z - 2 + \rho - K_d \frac{\lambda^2 D_0}{\nu_1^3} \frac{d - 2\sigma - 2 + 2\rho}{4d} \right) & \rho \geq 0, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{dD_0}{d\ell} &= D_0 (z - 2\alpha - d + 2\sigma) & \rho < \sigma \\ \frac{dD_1}{d\ell} &= D_1 \left( z - 2\alpha - d + 2\rho + \frac{K_d \lambda^2 D_1}{4 \nu_1^3} \right) & \rho \geq \sigma, \end{aligned} \quad (15)$$

and

$$\frac{d\lambda_\rho}{d\ell} = \lambda_\rho (\alpha + z - 2 + \rho). \quad (16)$$

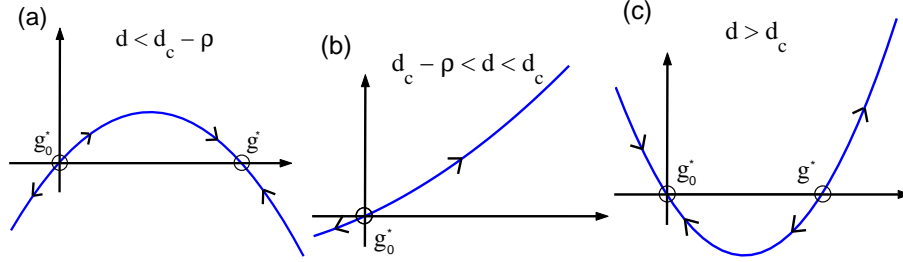
The last step is a discussion of the complete RG flow for the NKPZ equation. Four sectors in the  $\rho, \sigma$ -plane, in which solutions can be looked for, are identified. In the following, a detailed analysis in one of the sectors is presented, and results for the other sectors are provided. Sector I is defined by  $\rho \geq 0$  and  $\rho < \sigma$ . In this sector the flow equations are (14)b, (15)a and (16). As traditionally done (in Refs. [1, 17] for example), it is simpler to combine the flow equation into one equation for the coupling constant defined here as  $g \equiv K_d \lambda_\rho^2 D_0 / \nu_1^3 d$ . The RG flow of  $g$  becomes

$$\frac{dg}{d\ell} = (2 - d - \rho + 2\sigma) g + 3(d - 2\sigma - 2 + 2\rho) g^2, \quad (17)$$

and so the Fixed Points (FP) for  $g$  are

$$g_0^* = 0 \quad \text{and} \quad g^* = \frac{2 - d - \rho + 2\sigma}{3(2 - d - 2\rho + 2\sigma)}. \quad (18)$$

A special dimension comes out of the last expression, the so-called critical dimension which is  $d_c = 2 - \rho + 2\sigma$ . In Fig. 1 the RG flow of the coupling constant  $g$  for various dimensions is presented.



**Figure 1.** Coupling constant flow for the NKPZ equation in Sector I ( $\rho \geq 0$  and  $\rho < \sigma$ ), where  $d_c = 2 - \rho + 2\sigma$ . The three cases (a)-(c) cover all possible dimensions.

(a) The case  $d < d_c - \rho$ : As can be seen in Fig. 1(a), the nontrivial FP is the only stable FP in this region, and by plugging it into the flow equations gives the following scaling exponents  $\alpha = (2 - d - \rho + 2\sigma)/3$  and  $z = (d + 4 - 2\rho - 2\sigma)/3$ . These exponents are the generalizations of the exponents of the classical KPZ system with spatially correlated noise [17, 18, 19].

(b) For  $d_c - \rho < d < d_c$  (Fig. 1(b)), the trivial FP  $g_0^*$  is the only possible FP in the physical range as the nontrivial FP is negative. However,  $g_0^*$  is unstable, and so the system flows towards  $g = \infty$ . However, just like the strong coupling regime in KPZ, it is inaccessible to a perturbative consideration, and one can just indicate its existence without having a quantitative prediction for its scaling exponents.

(c) Last, as can be seen in Fig. 1(c), for  $d > d_c$  the trivial fixed point  $g_0^* = 0$  is stable, and the exponents are given by  $\alpha = (2 - d - \rho + 2\sigma)/2$  and  $z = 2 - \rho$ . These exponents correspond to the exponents of the Fractal Edwards-Wilkinson equation (i.e. a linear equation with a fractional Laplacian) with correlated noise. In addition, for a higher bare value of the coupling constant  $g > g^*$ , the system flows to  $g = \infty$ , which signals the appearance of the strong-coupling regime, again inaccessible to a perturbative approach. Thus, in this range of dimensions there is a possible phase transition between a weak-coupling to a strong-coupling regime.

The analysis of the possible phases in the other three sectors follows the same lines and the results are summarized in Table 1. Note that for the strong coupling phases, no quantitative result for the corresponding exponents is possible, apart from pointing out their existence.

As can be appreciated, the full description of the results given in Table 1 is quite rich. In order to gain more insight into this a special attention is given to the interesting

**Table 1.** A complete description of all the possible phases of the NKPZ problem using the modified DRG scheme for any value of  $d, \rho$  and  $\sigma$ . The first two columns give the scaling exponents  $\alpha$  and  $z$  for a particular phase, and the third column states each phase's validity condition. The values of the scaling exponents in the strong coupling phases are not accessible, and one can just indicate their existence. It is not even known whether they correspond to the same phase, and thus described by the same exponents or not.

$\alpha$	$z$	validity
$(2 - d - \rho + 2\sigma)/2$	$2 - \rho$ (FCEW)	$\rho \geq 0, \rho < \sigma, d > 2 - \rho + 2\sigma$
$(2 - d - \rho + 2\sigma)/3$	$(d + 4 - 2\rho - 2\sigma)/3$	$\rho \geq 0, \rho < \sigma, d < 2 - 2\rho + 2\sigma$
Strong Coupling - sector I		$\rho \geq 0, \rho < \sigma, d > 2 - 2\rho + 2\sigma$
$\frac{(2-d)(2-d+\rho)}{2(3-2d)}$	$2 - \rho - \frac{(2-d)(2-d+\rho)}{2(3-2d)}$	$\rho \geq 0, \rho \geq \sigma, d < 3/2$
Strong Coupling - sector II		$\rho \geq 0, \rho \geq \sigma, d > 3/2$
$(2 - d - \rho + 2\sigma)/2$	$2 - \rho$ (FCEW)	$\rho \geq 0, \rho \geq \sigma, d > 2 + \rho$
$(2 - d + 2\sigma)/2$	2 (CEW)	$\rho < 0, \rho < \sigma$
$(2 - d)/2$	2 (EW)	$\rho < 0, \rho \geq \sigma, d > 2$
Strong Coupling - sector IV		$\rho < 0, \rho \geq \sigma$

one dimensional case with  $\sigma \geq 0$ , where the following dynamic exponent is obtained:

$$z = \begin{cases} 2 & \rho < 0 \\ (3 - 3\rho)/2 & \rho \geq 0 \end{cases} . \quad (19)$$

Note that for  $\rho \geq 0$  the exact one-dimensional result is recovered [12], as was suggested in [11], and thus a major problem with DRG described above is solved.

To summarize, in this paper a modification of the classical DRG approach to systems out of equilibrium is presented in order to resolve problems with the results derived using traditional DRG for the Nonlocal KPZ equation. This approach extends beyond the NKPZ system, to any system out-of-equilibrium that produces under renormalization relevant terms which are not present in the original model. For the NKPZ system (as well as for the Nonlocal Molecular Beam equation [20] and the Fractal KPZ equation [21]) it is found that for certain values of the parameters a fractional Laplacian is generated under renormalization ( $\rho > 0$ ), or a correlated noise term ( $\rho > \sigma$ ). Thus, an inclusion of these terms in the original model (or put differently, by considering the right fixed-point dynamical system) leads to a correct description, and resolves the above-mentioned inconsistency with the exact  $1D$  result when  $\rho = 2\sigma$ .

The important lesson from this discussion is that in order to obtain a correct description of a given problem using DRG, one needs to verify that the correct fixed point dynamical system has been identified, and perform the perturbative expansion using this fixed-point system instead of the original model. Following this lesson can help to extend the range of applicability of DRG, since the direct contradiction with the exact result is settled.

An interesting application of these ideas could be the case of a driven wetting line of a fluid on a rough surface [13, 14] or the mathematically similar problem of an in-plane tensile crack propagating in a disordered material [15, 16] (a moving rather

than a pinned interface). In these problems, a long-range interaction term exists at the nonlinear order, and it is therefore vulnerable to similar difficulties. As suggested in Refs. [14, 16], it could be that the Edwards-Wilkinson system [2] is the relevant fixed-point system for the rough phase of these physical models. More work in that direction is needed to clarify this issue.

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